

# Power-law Behavior of High Energy String Scatterings in Compact Spaces

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**ABSTRACT.** We calculate high energy massive scattering amplitudes of closed bosonic string compactified on the torus. We obtain infinite linear relations among high energy scattering amplitudes. For some kinematic regimes, we discover that some linear relations break down and, simultaneously, the amplitudes enhance to power-law behavior due to the space-time T-duality symmetry in the compact direction. This result is consistent with the coexistence of the linear relations and the softer exponential fall-off behavior of high energy string scattering amplitudes as we pointed out previously. It is also reminiscent of hard (power-law) string scatterings in warped spacetime proposed by Polchinski and Strassler.

## 1. Introduction and Overview

It is well known that there are two fundamental characteristics of high energy string scattering amplitudes, which make them very different from field theory scatterings. These are the softer exponential fall-off behavior (in contrast to the hard power-law behavior of field theory scatterings) and the existence of infinite Regge-pole structure in the form factor of the high energy string scattering amplitudes.

For the last few years, high-energy, fixed angle behavior of string scattering amplitudes [1, 2, 3] was intensively reinvestigated for massive string states at arbitrary mass levels [4, 5, 6, 7, 8, 9, 10, 11, 12]. An infinite number of linear relations among string scattering amplitudes of different string states were discovered. An important new ingredient of these calculations is the zero-norm states (ZNS) [13, 14, 15] in the old covariant first quantized (OCFQ) string spectrum. The discovery of these infinite linear relations constitutes the *third* fundamental characteristics of high energy string scatterings, which is not shared by the usual point-particle field theory scatterings.

More recently, it was tempted to conjecture that [16, 17, 18] the newly discovered linear relations, or stringy symmetries, are responsible for the softer exponential fall-off string scatterings at high energies. One way to justify this conjecture (that is: the coexistence of the infinite linear relations and the softer exponential fall-off behavior of high energy string scatterings) is to find more examples of high energy string scatterings, which show the unusual hard power-law behavior and, simultaneously, give the breakdown of the infinite linear relations. With this in mind, in this report [19] we calculate high energy, fixed angle massive scattering amplitudes of closed bosonic string compactified on the torus [20]. In the Gross regime (GR), for each fixed mass level with given quantized and winding momenta  $(\frac{m}{R}, \frac{1}{2}nR)$ , we obtain infinite linear relations among high energy

scattering amplitudes of different string states. Moreover we discover that, for some kinematic regime, the so called Mende regime (MR), infinite linear relations with  $N_R = N_L$  break down and, simultaneously, the amplitudes enhance to power-law behavior [19]. It is the space-time T-duality symmetry that plays a role here.

There was another motivation to study the unusual high energy hard power-law behavior of string scattering. This is mainly motivated by the Gauge/String duality in the Type II B string theory on  $AdS_5$  background [21]. The work of Polchinski and Strassler and others [22, 23] suggested that the high energy behavior of string scattering in warped spacetime gives a consistent hard power-law behavior. It would be an interesting problem to understand the common features of the power-law string scatterings in these two different string backgrounds.

## 2. High Energy Scattering

We consider 26D closed bosonic string with one coordinate compactified on  $S^1$  with radius  $R$ . The closed string boundary condition for the compactified coordinate is

$$(1) \quad X^{25}(\sigma + 2\pi, \tau) = X^{25}(\sigma, \tau) + 2\pi Rn,$$

where  $n$  is the winding number. The momentum in the  $X^{25}$  direction is then quantized to be

$$(2) \quad K = \frac{m}{R},$$

where  $m$  is an integer. The left and right momenta are defined to be

$$(3) \quad K_{L,R} = K \pm L = \frac{m}{R} \pm \frac{1}{2}nR \Rightarrow K = \frac{1}{2}(K_L + K_R),$$

and the mass spectrum can be calculated to be

$$(4) \quad \begin{cases} M^2 = \left( \frac{m^2}{R^2} + \frac{1}{4}n^2R^2 \right) + N_R + N_L - 2 \equiv K_L^2 + M_L^2 \equiv K_R^2 + M_R^2, \\ N_R - N_L = mn \end{cases},$$

where  $N_R$  and  $N_L$  are the number operators for the right and left movers, which include the counting of the compactified coordinate. We have also introduced the left and the right level masses as

$$(5) \quad M_{L,R}^2 \equiv 2(N_{L,R} - 1).$$

In the center of momentum frame, the kinematic can be set up to be

$$(6) \quad k_{1L,R} = \left( +\sqrt{p^2 + M_1^2}, -p, 0, -K_{1L,R} \right),$$

$$(7) \quad k_{2L,R} = \left( +\sqrt{p^2 + M_2^2}, +p, 0, +K_{2L,R} \right),$$

$$(8) \quad k_{3L,R} = \left( -\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi, -K_{3L,R} \right),$$

$$(9) \quad k_{4L,R} = \left( -\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi, +K_{4L,R} \right)$$

where  $p \equiv |\tilde{p}|$  and  $q \equiv |\tilde{q}|$  and

$$(10) \quad k_i \equiv \frac{1}{2} (k_{iR} + k_{iL}),$$

$$(11) \quad k_i^2 = K_i^2 - M_i^2,$$

$$(12) \quad k_{iL,R}^2 = K_{iL,R}^2 - M_i^2 \equiv -M_{iL,R}^2.$$

With this setup, the center of mass energy  $E$  is

$$(13) \quad E = \frac{1}{2} \left( \sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) = \frac{1}{2} \left( \sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2} \right).$$

The conservation of momentum on the compactified direction gives

$$(14) \quad m_1 - m_2 + m_3 - m_4 = 0,$$

and T-duality symmetry implies conservation of winding number

$$(15) \quad n_1 - n_2 + n_3 - n_4 = 0.$$

The left and the right Mandelstam variables are defined to be

$$(16) \quad s_{L,R} \equiv -(k_{1L,R} + k_{2L,R})^2,$$

$$(17) \quad t_{L,R} \equiv -(k_{2L,R} + k_{3L,R})^2,$$

$$(18) \quad u_{L,R} \equiv -(k_{1L,R} + k_{3L,R})^2.$$

We now proceed to calculate the high energy scattering amplitudes for general higher mass levels with fixed  $N_R + N_L$ . With one compactified coordinate, the mass spectrum of the second vertex of the amplitude is

$$(19) \quad M_2^2 = \left( \frac{m_2^2}{R^2} + \frac{1}{4} n_2^2 R^2 \right) + N_R + N_L - 2.$$

We now have more mass parameters to define the "high energy limit". We are going to use three quantities  $E^2$ ,  $M_2^2$  and  $N_R + N_L$  to define different regimes of "high energy limit". The high energy regime defined by  $E^2 \simeq M_2^2 \gg N_R + N_L$  will be called Mende regime (MR). The high energy regime defined by  $E^2 \gg M_2^2$ ,  $E^2 \gg N_R + N_L$  will be called Gross region (GR). In the high energy limit, the polarizations on the scattering plane for the second vertex operator are defined to be

$$(20) \quad e^{\mathbf{P}} = \frac{1}{M_2} \left( \sqrt{p^2 + M_2^2}, p, 0, 0 \right),$$

$$(21) \quad e^{\mathbf{L}} = \frac{1}{M_2} \left( p, \sqrt{p^2 + M_2^2}, 0, 0 \right),$$

$$(22) \quad e^{\mathbf{T}} = (0, 0, 1, 0)$$

where the fourth component refers to the compactified direction. In the MR, we will use [19]

$$(23) \quad |N_{L,R}, q_{L,R}\rangle \equiv (\alpha_{-1}^{\mathbf{T}})^{N_L - 2q_L} (\alpha_{-2}^{\mathbf{P}})^{q_L} \otimes (\tilde{\alpha}_{-1}^{\mathbf{T}})^{N_R - 2q_R} (\tilde{\alpha}_{-2}^{\mathbf{P}})^{q_R} |0\rangle$$

as the second vertex operator in the calculation of high energy scattering amplitudes. The high energy scattering amplitudes in the MR can be calculated to be

$$\begin{aligned}
A \simeq & \left( -\frac{q \sin \phi (s_L + t_L)}{t_L} \right)^{N_L} \left( -\frac{q \sin \phi (s_R + t_R)}{t_R} \right)^{N_R} \left( \frac{1}{2M_2 q^2 \sin^2 \phi} \right)^{q_L + q_R} \\
& \cdot \left( \left( t_R - 2\vec{K}_{2R} \cdot \vec{K}_{3R} \right) + \frac{t_R^2 (s_R - 2\vec{K}_{1R} \cdot \vec{K}_{2R})}{s_R^2} \right)^{q_R} \\
& \cdot \left( \left( t_L - 2\vec{K}_{2L} \cdot \vec{K}_{3L} \right) + \frac{t_L^2 (s_L - 2\vec{K}_{1L} \cdot \vec{K}_{2L})}{s_L^2} \right)^{q_L} \\
(24) \quad & \cdot \frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)} B\left(-1 - \frac{t_R}{2}, -1 - \frac{u_R}{2}\right) B\left(-1 - \frac{t_L}{2}, -1 - \frac{u_L}{2}\right).
\end{aligned}$$

Eq.(24) is valid for  $E^2 \gg N_R + N_L$ ,  $M_2^2 \gg N_R + N_L$ .

**2.1. The infinite linear relations in the GR.** For the special case of GR with  $E^2 \gg M_2^2$ , Eq.(24) can be further reduced to

$$\begin{aligned}
\lim_{E^2 \gg M_2^2} A \simeq & \left( -\frac{2 \cot \frac{\phi}{2}}{E} \right)^{N_L + N_R} \left( -\frac{1}{2M_2} \right)^{q_L + q_R} E^{-1} \left( \sin \frac{\phi}{2} \right)^{-3} \left( \cos \frac{\phi}{2} \right)^5 \\
(25) \quad & \cdot \frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)} \exp \left( -\frac{t \ln t + u \ln u - (t + u) \ln(t + u)}{4} \right).
\end{aligned}$$

We see that, in the GR, for each fixed mass level with given quantized and winding momenta  $(\frac{m}{R}, \frac{1}{2}nR)$ , we have obtained infinite linear relations among high energy scattering amplitudes of different string states with various  $(q_L, q_R)$ . Note also that this result reproduces the correct ratios  $\left( -\frac{1}{2M_2} \right)^{q_L + q_R}$  obtained in the previous works [16, 17, 18]. However, the mass parameter  $M_2$  here depends on  $(\frac{m}{R}, \frac{1}{2}nR)$ .

**2.2. Power-law and breakdown of the infinite linear relations in the MR.** The power-law behavior of high energy string scatterings in a compact space was first suggested by Mende. Here we give a mathematically more concrete description. It is easy to see that the "power law" condition, i.e. Eq.(3.7) in Mende's paper [20]

$$(26) \quad k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} = \text{constant},$$

turns out to be

$$\begin{aligned}
& \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 + 2(\vec{K}_1 \cdot \vec{K}_2 + \vec{L}_1 \cdot \vec{L}_2) \\
(27) \quad & = \text{constant}.
\end{aligned}$$

As  $p \rightarrow \infty$ , due to the existence of winding modes in the compactified closed string, it is possible to choose  $(\vec{K}_1, \vec{K}_2; \vec{L}_1, \vec{L}_2)$  such that

$$(28) \quad \vec{K}_1 \cdot \vec{K}_2 + \vec{L}_1 \cdot \vec{L}_2 < 0,$$

and let  $(\vec{K}_1 \cdot \vec{K}_2 + \vec{L}_1 \cdot \vec{L}_2) \rightarrow -\infty$  to make

$$(29) \quad k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} \simeq \text{constant}$$

$$(30) \quad \Rightarrow s_L + s_R \simeq \text{constant}.$$

In our calculation, this condition implies the beta functions in Eq.(24) reduce to

$$(31) \quad B\left(-1 - \frac{t_R}{2}, -1 - \frac{u_R}{2}\right) B\left(-1 - \frac{t_L}{2}, -1 - \frac{u_L}{2}\right) \\ = \frac{\sin(\pi s_R/2) \Gamma(-\frac{t_R}{2} - 1) \Gamma(-\frac{u_R}{2} - 1) \Gamma(-\frac{t_L}{2} - 1) \Gamma(-\frac{u_L}{2} - 1)}{\pi^{\frac{s_R}{2}} (1 + \frac{s_R}{2}) (-1 + \frac{s_R}{2})},$$

which behaves as *power-law* in the high energy limit! On the other hand, it is obvious that the  $(q_L, q_R)$  dependent power factors of the amplitude in Eq.(24)

$$(32) \quad A_{q_L, q_R} \simeq \left( \frac{1}{2M_2 q^2 \sin^2 \phi} \right)^{q_L + q_R} \\ \cdot \left( \left( t_R - 2\vec{K}_{2R} \cdot \vec{K}_{3R} \right) + \frac{t_R^2 (s_R - 2\vec{K}_{1R} \cdot \vec{K}_{2R})}{s_R^2} \right)^{q_R} \\ \cdot \left( \left( t_L - 2\vec{K}_{2L} \cdot \vec{K}_{3L} \right) + \frac{t_L^2 (s_L - 2\vec{K}_{1L} \cdot \vec{K}_{2L})}{s_L^2} \right)^{q_L}$$

show *no* linear relations in the MR. Note that the mechanism to break the linear relations and the mechanism to enhance the amplitude to power-law are all due to  $E \simeq M_2$  in the MR. In our notation, Eq.(26) is equivalent to the following condition

$$(33) \quad \lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{\vec{K}_1 \cdot \vec{K}_2 + \vec{L}_1 \cdot \vec{L}_2} \sim \frac{E^2}{\left( \frac{m_1 m_2}{R^2} + \frac{1}{4} n_1 n_2 R^2 \right)} \sim -\mathcal{O}(1).$$

For our purpose here, as we will see soon, it is good enough to choose only one compactified coordinate to realize Eq.(33). First of all, in addition to Eq.(14) and Eq.(15), Eq.(4) implies

$$(34) \quad m_i n_i = 0, i = 1, 2, 3, 4 \text{ (no sum on } i \text{)}.$$

This is because three of the four vertex are tachyons. Also, since we are going to take  $n_2$  to infinity with fixed  $N_R + N_L$  in order to satisfy Eq.(33), we are forced to take  $m_2 = 0$ . In sum, we can take, say,  $m_i = 0$  for  $i = 1, 2, 3, 4$ , and  $n_1 = -n_2 = -n, n_3 = -2n, n_4 = 0$ , and then let  $n \rightarrow \infty$  to realize Eq.(33). Note that it is crucial to choose different sign for  $n_1$  and  $n_2$  in order to achieve the minus sign in Eq.(33). We stress that there are other choices to realize the condition. One notes that all choices implies

$$(35) \quad N_R = N_L.$$

It is obvious that one can also compactify more than one coordinate to realize the Mende condition. We conclude that the high energy scatterings of the "highly winding string states" of the compactified closed string in the MR behave as the unusual UV power-law,

and the usual linear relations among scattering amplitudes break down due to the unusual power-law behavior.

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